Mathematics 74-114 Midterm Examination - Solutions
Spring 2012

I. Let $p : \widetilde{X} \to X$ be a covering map which is n-sheeted, $2 \leq n \leq \infty$. Prove that there is no map $s : X \to \widetilde{X}$ such that $ps = \text{id}$. (Such a map is called a section of p.)

Proof 1

Let $x_0 \in X$. Since $ps = id$, $p_* s_* = id$, and so $p_* : \pi(\tilde{X}, s(x_0)) \to \pi(X, x_0)$ is onto. Let $\tilde{x}_0 = s(x_0)$, let $\tilde{x} \in p^{-1}(x_0)$ and let l be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}$. $[pl] = p_*[m]$ for some loop m in $\tilde{X}$ based at $\tilde{x}_0$. $\therefore$ $pl \sim pm$ so $l \sim m$.
$\tilde{x} = l(1) = m(1) = \tilde{x}_0$ $\therefore$ $p^{-1}(x_0)$ has one point. Contradiction
$\therefore$ $\nexists$ no such s.

Proof 2

$s : X \to \tilde{X}$ $p(sp) = (ps)p = p = p(id)$. $\therefore$ sp and id are both lifts of p. To show that they are equal, they must agree on a point. Let $x_0 \in X$ so $s(x_0) \in \tilde{X}$.
$sp(sx_0) = sx_0 = id(sx_0)$ and so $sp = id$. $\therefore$ p is a homeomorphism so $\tilde{X}$ is 1-sheeted. Contradiction

II. Let G be a group with unit e and let $S \subseteq G$ be a set. The **normal closure** $\overline{S}$ of S is defined to be the intersection of all normal subgroups of G which contain S. Prove

$$\overline{S} = \{e\} \cup \{c_1 \cdots c_k \mid k \geq 1,\ c_i = a_i s_i^{\epsilon_i} a_i^{-1}, \text{ where } a_i \in G,\ s_i \in S \text{ and } \epsilon_i = \pm 1\}.$$

Let $H = \{e\} \cup \{c_1 \cdots c_k\}$. Then H is closed under multiplication and inverses and so H is a subgroup. H is normal: Consider $x = a c_1 \cdots c_k a^{-1} = (ac_1a^{-1})(ac_2a^{-1}) \cdots (ac_ka^{-1})$. If $c_i = a_i s_i a_i^{-1}$ then $ac_i = (aa_i)s_i(aa_i)^{-1}$. $\therefore x \in H$ so H is normal. H contains S ($s_i = e s_i e^{-1}$) so H is a normal subgroup containing S. $\therefore \overline{S} \subseteq H$. Conversely, $\overline{S}$ is a normal subgroup containing S, so $\forall s_i \in S$, $c_i = a_i s_i a_i^{-1} \in \overline{S}$. $\therefore c_1 c_2 \cdots c_k \in \overline{S}$ $\therefore H \subseteq \overline{S}$. $\therefore H = \overline{S}$.

III. For any two based spaces (U, u_0) and (V, v_0) let $[U, V]$ denote the set of based homotopy classes of based maps $(U, u_0) \to (V, v_0)$. Now let (A, a_0), (X, x_0) and (Y, y_0) be based spaces and define

$$\theta : [A, X \times Y] \to [A, X] \times [A, Y]$$

by $\theta[f] = ([p_1 f], [p_2 f])$, where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the projections. Prove that θ is a well-defined bijection.

Suppose $\theta[f] = \theta[g]$ $\therefore$ $p_1 f \simeq p_1 g$ (homotopy f_t), $p_2 f \simeq p_2 g$ (homotopy g_t) Then $f \simeq g$ with homotopy (f_t, g_t) $((f_t, g_t)(a) = (f_t(a), g_t(a)))$. $\therefore$ θ is one-one. If $[h] \in [A, X]$ and $[k] \in [A, Y]$, then h and k determine $(h, k) : A \to X \times Y$ $((h,k)(a) = (h(a), k(a)))$ and $p_1(h,k) = h$, $p_2(h,k) = k$. $\therefore$ $\theta[(h,k)] = ([h], [k])$, so θ is onto.

IV. Let $f : X \to Y$ be a map and let $p : \widetilde{Y} \to Y$ be a covering map. Define the **pull-back** P by

$$P = \{(x, \widetilde{y}) \mid x \in X,\ \widetilde{y} \in \widetilde{Y} \text{ with } f(x) = p(\widetilde{y})\}.$$

Define maps $q : P \to X$ and $r : P \to \widetilde{X}$ by $q(x, \widetilde{y}) = x$ and $r(x, \widetilde{y}) = \widetilde{y}$.

1. Prove that $q : P \to X$ is a covering map.
2. Prove that r induces a bijection $q^{-1}(x) \to p^{-1}(f(x))$.
3. Prove that there is a section for $q : P \to X$ (that is, a map $s : X \to P$ such that $qs = \mathrm{id}$) if and only if f can be lifted to $\widetilde{Y}$.

1. Let $x \in X$ and let U be an elementary nbh of $f(x)$.
Claim: $f^{-1}(U)$ is an elementary nbh (of x) in X. :
$p^{-1}(U) = \bigcup V_j$, $q^{-1}(f^{-1}(U)) = r^{-1}p^{-1}(U) = \bigcup r^{-1}(V_j)$, a
union of disjoint open sets. Clearly $q' = q|_{r^{-1}V_j} : r^{-1}V_j \to$
$f^{-1}(U)$. q' is continuous and we show it is a homeo
by constructing an inverse, $k : f^{-1}(U) \to r^{-1}(V_j)$ defined
10 by $$k(x) = (x, (p|V_j)^{-1} f(x)).$$
$q'k(x) = x$ so $q'k = \mathrm{id}$. Let $(x, \tilde{y}) \in r^{-1}(V_j) \subseteq P$
$kq'(x, \tilde{y}) = k(x) = (x, (p|V_j)^{-1}(f(x)))$. But $\tilde{y} \in V_j$
and $fx = p\tilde{y}$ and so $\tilde{y} = (p|_{V_j})^{-1} fx$. $\therefore kq'(x, \tilde{y}) =$
$(x, \tilde{y})$, so $kq' = \mathrm{id}$. Since k is continuous, q' is a homeo.

2. $(x, \tilde{y}) \in q^{-1}(x)$, $fx = p\tilde{y}$ $r(x, \tilde{y}) = \tilde{y} \in p^{-1}(fx)$. $\therefore r$ induces
$r' : q^{-1}(x) \to p^{-1}(fx)$. We define $s' : p^{-1}(fx) \to q^{-1}(x)$: Given
6 $\tilde{y} \in p^{-1}(fx)$, $p\tilde{y} = fx$ so $(x, \tilde{y}) \in P$ and $q(x, \tilde{y}) = x$
Set $s'(\tilde{y}) = (x, \tilde{y})$ Then $r's' = \mathrm{id}$, $s'r' = \mathrm{id}$ so r' is bijection

3. If s is a section for q, rs is a lift of f to $\tilde{Y}$.
Conversely, if $\tilde{f}$ is a lift of f, define $s : X \to P$ by
4 $$s(x) = (x, \tilde{f}(x)).$$

V. Let $\widetilde{X}$ be any normal cover of X with covering map p, let $x_0 \in X$ be the base point and choose $\widetilde{x}_0 \in p^{-1}(x_0)$. Define $\theta : \pi(X, x_0) \to \mathcal{A}(\widetilde{X})$ (the group of deck transformations) as follows: Let $\alpha = [l] \in \pi(X, x_0)$ and let $\widetilde{l}$ be the lift of l to $\widetilde{X}$ starting at $\widetilde{x}_0$. Set $x_0' = \widetilde{l}(1)$. Then $p_*\pi(\widetilde{X}, \widetilde{x}_0)$ and $p_*\pi(\widetilde{X}, x_0')$ are conjugate, hence equal. Therefore there exists $\phi \in \mathcal{A}(\widetilde{X})$ with $\phi(\widetilde{x}_0) = x_0'$. Set $\theta(\alpha) = \phi$. Prove

1. θ is a homomorphism.
2. Kernel $\theta = p_*\pi(\widetilde{X}, \widetilde{x}_0)$.

Thus θ induces a homomorphism $\theta' : \pi(X, x_0)/p_*\pi(\widetilde{X}, \widetilde{x}_0) \to \mathcal{A}(\widetilde{X})$, where $\pi(X, x_0)/p_*\pi(\widetilde{X}, \widetilde{x}_0)$ is the set of right cosets. Prove

3. $\lambda\theta' = \mu$, where $\lambda : \mathcal{A}(\widetilde{X}) \to p^{-1}(x_0)$ and $\mu : \pi(X, x_0)/p_*\pi(\widetilde{X}, \widetilde{x}_0) \to p^{-1}(x_0)$ have been defined in class.

1. $\beta = [m]$, $\tilde{m}$ lift of m starting at $\tilde{x}_0$. Let $\psi \in \mathcal{A}(\tilde{X})$ such that $\psi(\tilde{x}_0) = \tilde{m}(1)$. $\varphi\tilde{m}$ is lift of m starting at $\varphi(\tilde{x}_0) = x_0'$
∴ Have path $\tilde{l} \cdot \varphi\tilde{m}$ in $\tilde{X}$ starting at $\tilde{x}_0$ and $p(\tilde{l} \cdot \varphi\tilde{m}) = l \cdot m$

8 $\theta(\alpha\beta) \in \mathcal{A}(\tilde{X})$ and $\theta(\alpha\beta)(\tilde{x}_0) = (\tilde{l} \cdot \varphi\tilde{m})(1) = \varphi(\tilde{m}1)$.
$\theta(\alpha)\theta(\beta) = \varphi\psi \in \mathcal{A}(\tilde{X})$ and $\varphi\psi(\tilde{x}_0) = \varphi(\tilde{m}1)$ ∴ $\theta(\alpha\beta) = (\theta\alpha)(\theta\beta)$.

2. Let $\gamma = [k] \in \operatorname{Ker}\theta$, ∴ $\theta(\gamma) = id$ Let $\tilde{k}$ be a lift of k starting at $\tilde{x}_0$. $id(\tilde{x}_0) = \tilde{x}_0 = \tilde{k}(1)$, so $\tilde{k}$ is a loop, $[\tilde{k}] \in \pi(\tilde{X}, \tilde{x}_0)$.
$p_*[\tilde{k}] = \gamma$ so $\gamma \in \operatorname{Im} p_*$. Conversely, if $\gamma = [k] = p_*[m]$

4 for m a loop in $\tilde{X}$ based at $\tilde{x}_0$, m is a lift of k. If $\theta(\gamma) = \varphi$, $\varphi(\tilde{x}_0) = m(1) = \tilde{x}_0$ ∴ $\varphi = id$ so $\gamma \in \operatorname{Ker}\theta$.

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3. Consider the diagram

$$\begin{tikzcd} \pi(X, x_0) \arrow[rr, "\theta"] \arrow[dr, "\bar{\mu}"'] & & \mathcal{A}(\tilde{X}) \arrow[dl, "\lambda"] \\ & p^{-1}(x_0) & \end{tikzcd}$$

$\lambda(\varphi) = \varphi(\tilde{x}_0)$. If $\alpha \in \pi(X, x_0)$, $\alpha = [l]$ and $\tilde{l}$ a lift of l

8 starting at $\tilde{x}_0$, $\bar{\mu}(\alpha) = \tilde{l}(1) = x_0'$. Then

$$\lambda\theta(\alpha) = \lambda(\varphi) = \varphi(\tilde{x}_0), \text{ where } \varphi \in \mathcal{A}(\tilde{X}) \text{ is}$$

such that $\varphi(\tilde{x}_0) = \tilde{l}(1) = x_0'$.

$$\therefore \lambda\theta(\alpha) = \varphi(\tilde{x}_0) = x_0' = \bar{\mu}(\alpha)$$

$\therefore$ The diagram is commutative: $\lambda\theta = \bar{\mu}$. If $\nu : \pi(X, x_0) \longrightarrow \pi(X, x_0)/p_*\pi(\tilde{X}, \tilde{x}_0)$ is the quotient map $\bar{\mu} = \mu\nu$ and $\theta'\nu = \theta$. ~~Therefore~~ $\therefore$ $\lambda\theta = \bar{\mu}$ becomes $\lambda\theta'\nu = \mu\nu$. Since ν is onto, $\lambda\theta' = \mu$.